TRINITY COLLEGE

Year 11 Mathematics Specialist
Test 3 2019

Calculator Assumed (Scientific Calculator only)
Proof

STUDENT'S NAME SOLUTIONS

DATE: Monday 20 May **TIME:** 50 minutes **MARKS:** 50

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Consider the statement: If n is prime, then n is odd.

(a) State the converse of the statement and comment on its validity. [2]

- If n is odd, then n is prime
- False

(b) State the inverse of the statement and comment on its validity. [2]

- If n is not prime then n is not odd
- False

(c) State the contrapositive of the statement and comment on its validity. [2]

- If n is not odd then n is not prime
- False

2. (5 marks)

For each of the following, decide whether the conjecture is true or false. If it is true, prove it. If it is false, give a counter example.

(a) The sum of 3 consecutive even numbers is never a multiple of 3. [2]

counter example : $2+4+6=12$

12 is a multiple of 3

$\therefore$ conjecture is false

(b) When the product of two consecutive odd numbers is added to the next consecutive odd number, the result is always even. [3]

$= (2m-1)(2m+1) + (2m+3)$

$= 4m^2 - 1 + 2m + 3$

$= 4m^2 + 2m + 2$

$= 2(2m^2 + m + 1)$

$\downarrow$

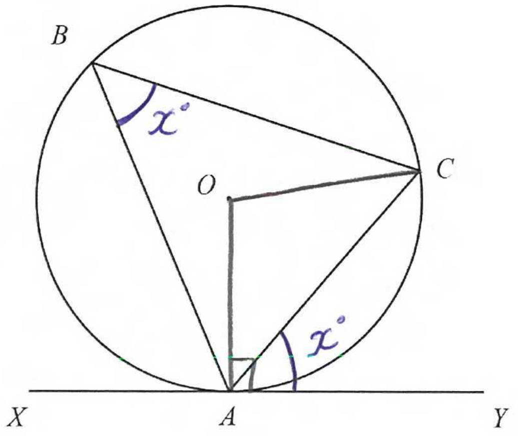
any number multiplied by 2 results in an even number

$\therefore$ conjecture is true

5. (4 marks)

Prove the Alternate Segment Theorem

i.e. for the circle below, centre O , prove $\angle CAY = \angle ABC$



Let $\angle CAY = x$

$\bar{OA}$ and $\bar{OC}$ are both radii

$\angle OAY = 90^\circ$ (tangent)

$\angle OAC = 90 - x$

$\angle OCA = 90 - x$ (isosceles Δ)

$\angle COA = 180 - 2(90 - x)$ (angle sum in Δ)

$= 2x$

$\angle ABC = x$ (angle at centre double angle at circumference)

$\therefore \angle CAY = \angle ABC$ Q.ED

3. (3 marks)

For a group of 25 people, prove that there must be at least one day of the week on which at least 4 of those people were born. Justify your answer using the pigeon hole principle.

Pigeon holes = days of the week Pigeons = people

If 21 people were chosen the worst case scenario is that 3 people are born on each 7 days. When the 22nd person is chosen at least one day must have 4 people.

$25 > 22$ $\therefore$ must be a day with at least 4 people.

4. (3 marks)

From the set of counting numbers up to one hundred {1, 2, 3, 4, …, 99, 100}, a subset of k numbers is chosen. What is the smallest number that k can be, such that there must be a pair of numbers in the subset that add to an odd number? Justify your answer using the pigeon hole principle.

Pigeon holes = odd/even Pigeons = counting numbers

To add to an odd number, there must be an odd and even number.

The first 50 numbers chosen could be all even or all odd.

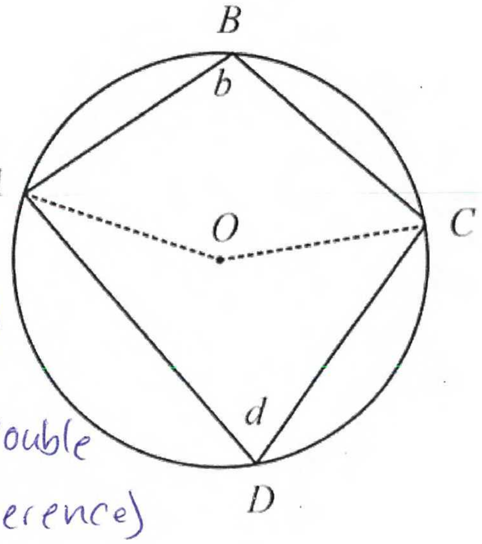
$\therefore$ k must be 51

6. (5 marks)

Use proof by contradiction to prove that the opposite angles of a cyclic quadrilateral are supplementary.

let $\angle ABC = b$ & $\angle ADC = c$

Assume $b + d \neq 180°$



$\angle AOC$ (obtuse) $= 2d$ (angle at centre double angle at circumference)

$\angle AOC$ (reflex) $= 2b$ (angle at centre double angle at circumference)

$\angle AOC$ (obtuse) $+ \angle AOC$ (reflex) $= 360°$ (angle at point)

$2d + 2b = 360°$

$d + b = 180$

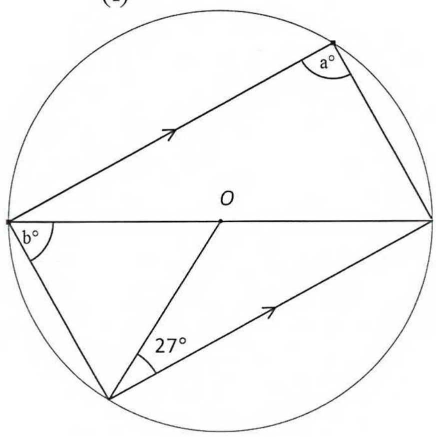
This contradicts our original assumption

$\therefore$ conjecture is true.

7. (10 marks)

(a) Determine the value of the pronumeral in each of the following.

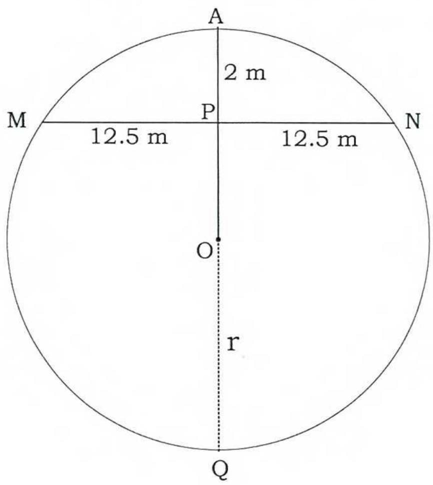
(i) [2]



$a = 90°$

$b = 63°$

(ii) r = radius of the circle [3]



$12.5^2 = (2r-2) \times 2$

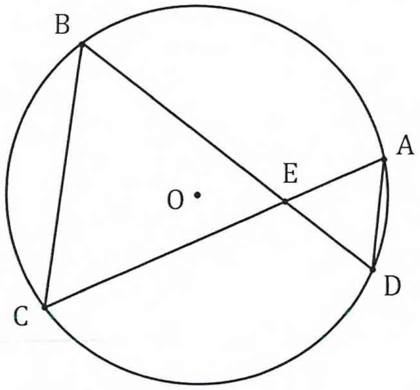
$156.25 = 4r - 4$

$160.25 = 4r$

$40.06m = r$

(b) In the circle with centre O drawn below, chord AC intersects chord BD at E.

(i) Explain, with reasoning, why triangles AED and BEC are similar. [3]



$\angle CBD = \angle CAD$ (angles on same arc)

$\angle BCA = \angle BDA$ (angles on same arc)

$\angle BEC = \angle AED$ (vertically opposite)

$\therefore \triangle AED \sim \triangle BEC$ (AAA)

(ii) Prove that when two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord. [2]

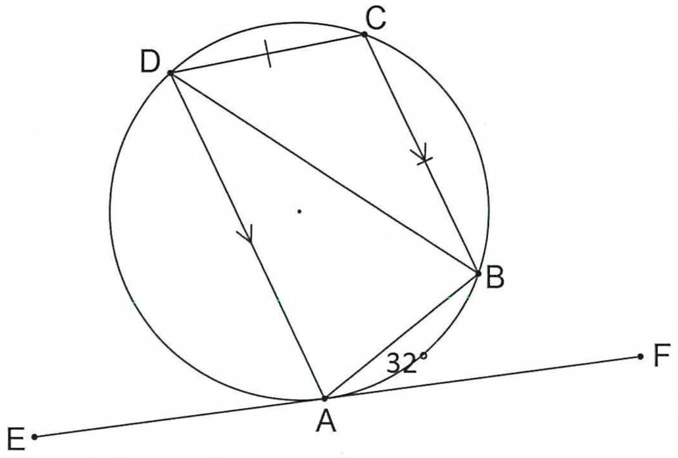
$$\frac{BE}{AE} = \frac{CE}{DE}$$

Hence, $BE \times DE = CE \times AE$

8. (4 marks)

In the diagram below, ABCD is a cyclic quadrilateral.

- EAF is a tangent to the circle at A.
- BC = CD.
- AD is parallel to BC.
- ∠BAF = 32°.



Determine, with **reasons**, the size of ∠BAD.

∠ADB = 32° (angle in alternate segment)

∠DBC = 32° (alternate angles in parallel lines)

∠BCD = 32° (isosceles Δ)

∠BCD = 180 − 2(32) (angle sum in Δ)

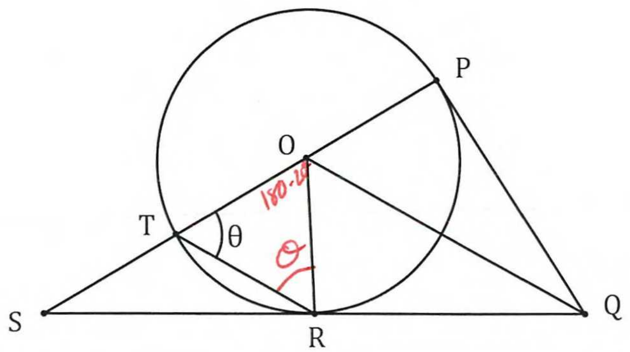
= 116°

∠BAD = 180 − 116 (opposite angles in cyclic quad)

= 64°

9. (10 marks)

In the diagram below, POT is a diameter of circle with centre O, QP is a tangent to the circle at P, QR is a tangent to the circle at R and PT is extended to meet QR extended at S. Hint: Let $\angle OTR = \theta$.



(a) Prove that ΔOPQ is congruent to ΔORQ. [3]

$\overline{OP} = \overline{OR}$ (radii)

$\overline{OQ} = \overline{OQ}$ (common)

$\angle OPQ = \angle ORQ$ (tangent - radius angle)

$\therefore \Delta OPQ \equiv \Delta ORQ$ (RHS)

(b) Prove that OQ is parallel to TR. [4]

$\overline{OT} = \overline{OR}$ (radii)

$\therefore \angle OTR = \angle ORT = \theta$ (isosceles Δ)

$\angle TOR = 180 - 2\theta$ (angle sum in Δ)

$\angle POR = 180 - (180 - 2\theta)$ (supplementary $\angle$'s)

$= 2\theta$

$\angle POQ = \angle ROQ = \theta$ (congruent Δ)

$\angle RTO = \angle QOP = \theta$ (corresponding $\angle$'s)

Hence, OQ is parallel to TR

(c) If $TR = TS$, deduce that ΔOTR is equilateral. [3]

ΔTSR is isosceles (given)

$\angle STR = 180 - \theta$ (supplementary $\angle$)

$\angle TSR = \angle TRS = \frac{180-(180-\theta)}{2}$ (angle sum)

$= \frac{\theta}{2}$

$\angle TRS = 90 - \theta$ (tangent - radius angle)

$90-\theta = \frac{\theta}{2}$

$\theta = 60^\circ$

$\therefore \angle OTR = \angle ORT = \angle TOR = 60^\circ$

Hence, ΔOTR is equilateral